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BIOGRAPHY.

BOLYAI FARKAS. [WOLFGANG BOLYAI.]

BY DR. GEORGE BRUCE HALSTED.

FOR the treatment of parallels, what Frischauf calls "das anschaulichste Axiom," is due to the researches of Bolyai Farkas. He gives it in his "Kurzer Grundriss eines Versuchs" etc., p. 46, as follows: "Koennten jede 3 Punkte, die nicht in einer Geraden sind, in eine Sphaere fallen; so waere das Eucl. Ax. XI. bewiesen." Thus the space whose every three points are co-straight or concyclic is Euclidean.

But in his Autobiography written in Magyar, of which my forthcoming life of the Bolyais contains the first translation ever made, he says: "Yet I was not satisfied with my attempts to prove the Problem of Parallels, which was ascribable to the long discontinuance of my studies, or more probably it was due to myself that I drove this problem to the point which robbed my rest, deprived me of tranquility."

Hitherto what was known of the Bolyais came wholly from the published works of the father, Bolyai Farkas, and from a brief article by Architect Fr. Schmidt of Budapest, "Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya. Grunerts Archiv, Bd. 48, 1868, p. 217.

In two communications sent me in September and October, 1895, Herr Schmidt has very kindly and graciously put at my disposal the results of his subsequent researches which I will here reproduce. But meantime I have from entirely another source come most unexpectedly into possession of original documents so extensive, so precious that I have determined to issue them in a

BOLYAI FARKAS. [WOLFGANG BOLYAI.]

separate volume devoted wholly to the life of the Bolyais ; but these are not used in the sketch here given.

Bolyai Farkas was born February 9th, 1775, at Bolya in that part of Transylvania (Erdély) called Székelyföld. He studied first at Enyed, afterward at Klausenburg (Kolozsvár), then went with Baron Simon Kemény to Jena and afterward to Goettingen. Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life.

The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss. Every-one who met Bolyai felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written in 1801 that Bolyai was one of the most extraordinary men he had ever known.

He returned home in 1799, and in January, 1804, was made professor of mathematics in the Reformed College of Maros-Vásárhely. Here for 47 years of active teaching he had for scholars nearly all the professors and nobility of the next generation in Erdély.

Sylvester has said that mathematics is poesy.

Bolyai's first published works were dramas.

His first published book on mathematics was an arithmetic : Az arithmetica eleje. 8vo. I—XVI, 1—162 pp. The copy in the library of the Reformed College is enriched with notes by Bolyai János.

Next followed his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, 8vo. with title as follows: TENTAMEN | JUVENTUTEM STUDIOSAM | IN ELEMENTA MATHESEOS PURÆ, ELEMENTARIS AC | SUBLIMIORIS, METHEDO INTUITIVA, EVIDENTIA— | QUE HUIC PROPRIA, INTRODUCENDI. | CUM APPENDICE TRIPLICE. |

Auctore Professore Matheseos et Physices Chemiæque | Publ. Ordinario. | Tomus Primus. | *Maros Vásárhelyini.* 1832. | Typis Collegii Reformatorum per JOSEPHUM, et | SIMEONEM KALI de felsö Vist. | At the back of the title : Imprimatur. | M. Vásárhelyini Die | 12 Octobris 1829. |

The now world renowned Appendix by Bolyai János was an afterthought of the father, who prompted the son not 'to occupy himself with the theory of parallels,' as Staeckel says, but to translate from the German into Latin a condensation of his treatise, of which the principles were discovered and properly appreciated in 1823, and which was given in writing to J. W. von Eckwehr in 1825.

The father, without waiting for Vol. II., inserted this Latin translation, with separate paging (1—26), as an Appendix to his Vol. I., where, counting a page for the title and a page 'Explicatio signorum,' it has twenty-six numbered pages, followed by two unnumbered pages of Errata.

The treatise itself, therefore, contains only twenty-four pages—the most extraordinary two dozen pages in the whole history of thought !

Milton received but a paltry 5 pounds for his Paradise Lost ; but it was at least plus 5. Bolyai Janos, as we learn from Vol. II., p. 384 of '*Tentamen*,' contributed for the printing of his eternal 26 pages, 104 florins 54 kreuzers.

That this Appendix was finished considerably before the Vol. I., which it follows, is seen from the references in the text, breathing a just admiration for the Appendix and the genius of its author,

Thus Bolyai Farkas says, p. 452 : Elegans est conceptus *similium*, quem J. B. *Appendicis Auctor* dedit; again, p. 489: *Appendicis Auctor*, rem acumine singulari aggressus, Geometriam pro omni casu absolute veram posuit; quamvis e magna mole, tantum summe necessaria, in Appendice hujus tomi exhibuerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisitionibus elegantibus) brevitatis studio omissis. And the volume ends as follows, p. 502: Nec operae pretium est plura referre ; quum res tota ex altiori contemplationis puncto, in ima prenetranti oculo, tractetur in Appendice sequente, a quovis fideli veritatis purae alumno digna legi.

The father gives a brief resumé of the results of his own determined, life-long, desperate efforts to do that at which Saccheri, J. H. Lambert, Gauss also had failed, to establish Euclid's theory of parallels *a priori*.

He says, p. 490: "tentamina idcirco quae olim feceram, breviter exponenda veniunt ; ne saltem alius quis operam eandem perdat." He anticipates J. Delboeuf's "Prolégomènes philosophiques de la géométrie et solution des postulato," with the full consciousness in addition that it is *not* the solution,—that the final solution has crowned not his own intense efforts, but the genius of his son.

This son's Appendix which makes all preceding space only a special case, only a species under a genus, and so requiring a descriprive adjective, *Euclidean*, this wonderful production of pure genius, this strange Hungarian flower was saved for the world after more than thirty-five years of oblivion, by the rare erudition of Professor Richard Baltzer of Dresden, afterward professor in the University of Giessen. He it was who first did justice publicly to the works of Lobachevski and Bolyai.

Incited by Baltzer, 1866, J. Hoüel issued a French translation of Lobachevski's Theory of Parallels and in a note to his Preface says: "M. Richard Baltzer, dans la seconde édition de ses excellents *Éléments de Géométrie*, a, le premier, introduit ces notions exactes à la place qu'elles doivent occuper." Honor to Baltzer ! But alas ! father and son were already in their graves !

Fr. Schmidt in the article cited (1868) says: "It was nearly forty years before these profound views were rescued from oblivion, and Dr. R. Baltzer, of Dresden, has acquired imperishable titles to the gratitude of all friends of science as the first to draw attention to the works of Bolyai, in the second edition of his excellent Elemente der Mathematik (1866-67). Following the steps of Baltzer, Professor Hoüel, of Bordeaux, in a brochure entitled: Essai critique sur les principes fondamentaux de la Géométrie élémentaire, has give extracts from Bolyai's book, which will help in securing for these new ideas the justice they merit."

The father refers to the son's Appendix again in a subsequent book, Űrtan elemei Kezdőknek [Elements of the science of space for beginners] (1850-51), pp. 48. In the College are preserved three sets of figures for this book, two by the

author, and one by his grandson, a son of János. The last work of Bolyai Farkas, the only one composed in German, is entitled: Kurzer Grundriss eines Versuchs

I. Die Arithmetik, durch zvekmässig Konstruirte Begriffe, von eingebildeten und unendlich-kleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen.

II. In der Geometrie, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit u.d.gl. nicht nur scharf zu bestimmen; sondern auch ihr Seyn im Raume zu beweisen: und da die Frage, *ob zwey von der dritten geschnittene Geraden, wenn die summe der inneren Winkel nicht* $=2R$, *sich schneiden oder nicht?* Niemand auf der Erde ohne ein Axiom (wie Euklid das XI) aufzustellen, beantworten wird; die davon unabhängige Geometrie abzusondern; und eine auf die *Ja*-Antwort, andere auf das *Nein* so zu bauen, das die Formeln der letzten, auf ein Wink auch in der ersten gültig seyen.

Nach ein lateinischen Werke von 1829, M. Vásárhely, und eben daselbst gedruckten ungrischen.

Maros Vásárhely 1851. 8vo. pp. 88.

In this book he says, referring to his son's Appendix: "Some copies of the work published here were sent at that time to Vienna, to Berlin, to Goettingen. . . . . From Goettingen the giant of mathematics, who from his pinnacle embraces in the same view the stars and the abysses, wrote that he was surprised to see accomplished, what he had begun, only to leave it behind in his papers." This refers to 1832. The only other record that Gauss ever mentioned the book is a letter from Gerling written October 31st, 1851, to Wolfgang Bolyai on receipt of a copy of 'Kurzer Grundriss.' Gerling, a scholar of Gauss, had been from 1817 Professor of Astronomy at Marburg. He writes: "I do not mention my earlier occupation with the theory of parallels, for already in the year 1810—1812 with Gauss, as earlier as 1809 with J. F. Pfaff I had learned to perceive, how all previous attempts to prove the Euclidean axiom had miscarried. I had then also obtained preliminary knowledge of your works, and so, when I first [1820] had to print something of my view thereon, wrote it exactly so, as it yet stands to read on page 187 of the latest edition.

We had about this time [1819] here a law professor Schweikart, who was formerly in Charkow, and had attained to similar ideas, since without help of the Euclidean axiom he developed in its beginnings a geometry which he called Astralgeometry. What he communicated to me thereon, I sent to Gauss, who then informed me, how much farther already had been attained on this way and later also expressed himself about the great acquisition, which is offered to the few expert judges in the Appendix to your book."

The 'latest edition' mentioned appeared in 1851, and the passage refarred to is: "This proof [of the parallel-axiom] has been sought in manifold ways by acute mathematicians, but yet until now not found with complete sufficiency. So long as it fails, the theorem, as all founded on it, remains a hypothesis, whose

validity for our life indeed is sufficiently proven by *experience*, whose *general, necessary exactness* however could be doubted without absurdity."

Alas! that this feeble utterance should have seemed sufficient for more than thirty years to the associate of Gauss and Schweikart, the latter certainly one of the independent discoverers of the non-Euclidean geometry. But then since neither of these sufficiently realized the transcendent importance of the matter to publish any of their thoughts on the subject, a more adequate conception of the issues at stake could scarcely be expected of the scholar and colleague. How different with Bolyai János and Lobachévski, who claimed at once, unflinchingly, that their discovery marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or of science, demonstrating as had never been proven before the supremacy of pure reason at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

Austin, Texas, December 16th, 1895.

THE DUPLICATION OF THE NOTATION FOR IRRATIONALS.

By JOS. V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

Authorities agree in crediting Rudolff (1525), the German cossist, with the introduction of the radical sign, $\surd$, not precisely as we use it, but one such mark for a square root, three for a cube, and two for a fourth root. Cantor thinks it probably originated from a West-Arabian custom of using dots, by makings *lines* of the dots, and connecting them in the making by lighter lines. These dots in turn originated, it is thought, in the use of the letter, dschim, the first in the Arabian word for *root*. Rudolff was followed by Stifel in the employment of this notation, and afterwards Girard (1633) changed it to the present form. By the middle of the 17th century the mark had come into general use. The exponent notation, though first used by Rudolff and Stifel in a crude form, was employed as we now have it for integral values of the exponents by Descartes. Soon after, Wallis, in his *arithmetica infinitorum* (1656), interpreted and used the simpler forms of fractional exponents, though Stevin (1585) had suggested the meaning to be assigned them. Then in 1676 Newton wrote to Oldenburg "since algebraists write a^2, a^3, a^4, etc., for aa, aaa, $aaaa$, etc., so I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, for $\surd a$, $\surd a^3$, $\surd e.a^5$." Newton went further in connection with his binomial theorem, and generalized this use of exponents into the exponential function. The question naturally arises why was it that the old notation for roots was not replaced by the new as had been done in numerous instances before? Doubtless the best

reason for this is the fact that the radical signs were firmly intrenched by extended use before the fractional exponents as we have them were even thought of.

Now from one standpoint at least this duplication of marks for one of the commonest operations in mathematics is unfortunate. It certainly complicates unnecessarily a rather difficult part of elementary algebra. Doubtless all would agree that one or the other should be given up unless there is a good and sufficient reason for its retention. If either is to be discarded there is no question for a moment as to which should go. The use of fractional exponents is in perfect accord with that of integral ones, and introduces no new marks or conventions, while the radical sign notation is out of harmony with everything else in the algebraic notation. The radical sign and index are new marks, while the fractional exponent is an old quantity in a new place whose interpretation is quite natural. However, it should be said that the fractional exponent notation is ambiguous, since, in general,

$\left(a^m\right)^{\frac{1}{n}}$ will not be the same as $\left(a^{\frac{1}{n}}\right)^m$, though each reduces to $a^{\frac{m}{n}}$. Nevertheless, even here the fractional exponent notation is to be preferred to the others, since the elementary treatment of irrationals virtually depends on the ignoring of this difference. (See, for example, Todhunter's Algebra, ed. 1877, p. 153; Chrystal's Treatise, Chapter X, Part II.) Not a few authors succeed by their manner of treatment in slurring this over. In this connection it ought to be said that some authors' books show distinct traces of their having been confused by the double surd notation. If authors themselves are not clear in their treatment of irrationals, it is likely that their students also will be more or less puzzled. This of itself would be a sufficient justification of an effort to remove the difficulty.

One obstacle in the way of dispensing entirely with the radical signs consists in the practical difficulty of writing and printing fractional exponents. But this, one is constrained to believe, can readily be overcome. And first it may be remarked that there is the same justification for omitting the numerator 1 in a fractional exponent that there is for never writing the integral exponent 1. When omitted it can be understood. Then again there is the same justification for dropping the denominator 2 in the exponent that there is for understanding the radical index 2 when no index is written. Thus all that is left of the fractional exponent ½ is the horizontal line or the solidus oblique line. To make the changes suggested clear to the reader, some expressions are written below with their values in the three notations:

RADICAL NOTATION.		FRAC. EXPONENT NOT.		PROPOSED NOTATION.
$2\sqrt{a}$	$=$	$2a^{\frac{1}{2}}$	$=$	$2a^{/}$ *

*The marks for primes would differ from this sign in being shorter and vertical. However, it would be better to write subscripts in place of them.

$$3\sqrt[3]{26} \quad = \quad 3(26)^{3} \quad = \quad 3(26)^{/3}$$

$$\sqrt{a+m} \quad = \quad (a+m)^{\frac{1}{2}} \quad = \quad (a+m)^{/}$$

$$\sqrt[4]{\frac{a^2+b^2+c^2}{2abc}} \quad = \quad \left(\frac{a^2+b^2+c^2}{2abc}\right)^{\frac{1}{4}} \quad = \quad \left(\frac{a^2+b^2+c^2}{2abc}\right)^{/4}$$

$$\sqrt[5]{(x^3+3xy^2)^3} \quad = \quad (x^3+3xy^2)^{\frac{3}{5}}\text{, or } (x^3+3xy^2)^{3/5}$$

The proposed notation would do away with vinculums and would use preferably the solidus sign for division as is the tendency now in English mathematical and scientific books. In printing, $\surd$ would be replaced by $)^{/}$ on one type, and in script the latter would be made, without lifting the pen, in loop form. However, when the numerator of the fractional exponent is other than unity, the usual fractional exponent notation (which for this case is preferable to the radical sign notation) would be employed. Notice that by the simple changes proposed, which are perfectly natural ones, all the advantages of the duplicate notation would be preserved with none of its disadvantages, such as the use of the unsightly hieroglyphic-like radical sign (giving as it does a forbidding appearance to the printed page), and the confusion which arises from the simultaneous use of two distinct notations for the same operation.

In conclusion it should be emphasized that mathematicians themselves are not likely to feel the need or approve of any change in the algebraic notation. Like the reform in spelling, it is in the interest chiefly of the hundreds of thousands of students of elementary mathematics yet to come, and not in that of those who have already mastered the two notations, that this reform is urged. Surely it is not too much to ask that the fractional exponents as now written be employed exclusively (instead of largely as now) in all higher works involving the use of algebraical symbols. The abridgments would then be likely to come as a matter of course.

Stevens Point, Wisconsin, May 11, 1895.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from December Number.]

The Construction of Non-Primitive Groups with Three Systems of Non-Primitivity.

Let the degree of the required group G be $3n$. G must be a subgroup (using subgroup in its broad sense in which it includes the group itself and identity) of

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}.$$

If G_1 is not identity,* its constituents must be conjugate transitive subgroups of these three systems.

If we designate the systems by A, B, and C, the permutations of the systems must correspond to a group of these three letters, for if these permutations would not form a group of operations G itself could not be a group. Hence every non-primitive group with three systems must correspond to one of the following groups :

$$(ABC)\text{cyc} \qquad (ABC)\text{all}$$

Since the former of these is a subgroup of the latter it follows that at least a part of every non-primitive group in three systems corresponds to

$$(ABC)\text{cyc}$$

we proceed to find this part. By a course of reasoning similar to that employed under two systems it follows that all the substitutions which transform any G_1 according to ABC must be contained in

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}\ a_1 b_1 c_1 . a_2 b_2 c_2 \ldots\ldots a_n b_n c_n$$

and all those which transform G_1 according to ABC must be contained in

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(c_1 c_2 \ldots\ldots c_n)\text{all}\ a_1 c_1 b_1 . a_2 c_2 b_2 \ldots\ldots a_n c_n b_n$$

These sets are not independent, for if

$$s_\gamma \qquad \gamma = 1, 2, \ldots\ldots (n!)^3$$

represents the substitution of one set then will the $(n!)^3$ different corresponding values of

$$s_\gamma^{-1}$$

represent the substitutions of the other set.

If in any non-primitive group G_2 stands for the substitution belonging to the first set and G_3 for those belonging to the second set, and if g_2 and g_3 represent the number of substitutions in G_2 and G_3 respectively we derive from the fact that if a group contains s_γ it must also contain s_γ^{-1} that

$$g_2 = g_3$$

If in any non-primitive group we multiply any substitution of G_2 by all

*This case was not considered under two systems of non-primitivity. It was unnecessary to consider it. For, since a transitive group contains substitutions which replace a given letter by all of the letters involved it follows that the order of a non-primitive group is always equal to its degree. It can easily be shown that the order of any transitive group is a multiple of its degree.

the substitutions of G_3 we obtain g_3 different substitutions of G_1, hence

$$g_1 \geqq g_3.$$

If we multiply a given substitution of G_3 into all the substitutions of G_1 we obtain g_1 different substitutions of G_3, hence

$$g_3 \geqq g_1.$$

Combining the last two relations with the preceding we obtain for any non-primitive group with three systems of non-primitivity

$$g_1 = g_2 = g_3.$$

Since the relation between G_2 and G_3 is such that we can derive one directly from the other we shall generally consider only G_2. But G_2 can be directly obtained from G_1 provided we have given one of the substitutions of G_2. Hence to construct the non-primitive group (or the part of a non-primitive group) corresponding to

$$(ABC)\text{cyc}$$

it is only necessary to find G_1 and one substitution (s_γ) corresponding to ABC.

s_γ must clearly satisfy the following conditions:

(1) Its cube is found in G_1.

(2) It transforms G_1 into itself.

(3) It permutes the systems according to ABC.

These three conditions are sufficient for if any substitution s_γ fulfills these conditions then is

$$G_1 + G_1 s_\gamma + G_1 s_\gamma^{-1}$$

a non-primitive group for

$$G_1 s_{\gamma_1} G_1 = G_1 s_{\gamma_1} G_1 s_\gamma^{-1} s_{\gamma_1} = G_1 s_{\gamma_1}$$

$$G_1 s_\gamma^{-1} G_1 - G_1 s_\gamma^{-1} G_1 s_{\gamma_1} s_\gamma^{-1} = G_1 s_\gamma^{-1}$$

etc., etc., etc.

It remains to prove that the three given conditions are necessary as well as sufficient, i. e., we have to show that none of the three pair of conditions is sufficient. The pair which excludes the last condition is evidently insufficient, and the following examples prove that the other two pair are also insufficient.

1	1	1	1
abc	def	ghi	$abc.def.ghi$
acb	dfe	gih	$acb.dfe.gih$
			$ab.de.gh$
			$ac.df.gi$
			$bc.ef.hi$

For *aehbdg.cfi* satisfies the second and third but not the first of the three conditions if we take the first of these groups for G_1, and *aehbficdg* satisfies the first and third but not the second if we take the second of these groups for G_1. Hence we see that the three given conditions are necessary as well as sufficient.

If the transitive constituents of G_1 admit only a cyclical (not a symmetric) permutation then it is impossible to construct a G corresponding to (ABC)all and involving the given G_1. If they admit a symmetric permutation we have to add to the part of G corresponding to (ABC)cyc sufficient substitutions to make it correspond to (ABC)all. By a course of reasoning similar to that which we have just pursued we prove that it is only necessary to find one substitution s_β corresponding to AB, and that s_β must satisfy the following conditions:

(1) Interchange the first two systems.

(2) Have its square in G_1.

(3) Transform the group corresponding to ABC into itself.

To fix these ideas we proceed to the construction of the non-primitive groups of degree six which contain three systems of non-primitivity. We shall then have found all the non-primitive groups up to degree eight as no such groups can exist for degree seven, or any other prime degree.

Non-Primitive Groups of Degree Six with Three Systems of Non-Primitivity.

G_1 must be one of the following four groups: $(ab)(cd)(ef)$, $\{(ab)(cd)(ef)\}$pos, $(ab.cd.ef)$, 1 G_2 must be contained in

$$(ab)(cd)(ef)\ ace.bdf$$

(*a*) If $G_1=(ab)(cd)(ef)$ then will *ace.bdf* evidently satisfy the three necessary conditions, we thus obtain a non-primitive group corresponding to ABC, whose order is 24, viz:

$$(1)\qquad (ab)(cd)(ef)\ (ace.bdf)\text{cyc}=(abcdef)_{24_5}{}^*$$

For s_β we may take *ac.bd*. This leads to a group of order 48 which has the preceding group as a self-conjugate sub-group. The group is

$$(2)\qquad (ab)(cd)(ef)(ace.bdf)\text{cyc}(ac.bd)=(abcdef)_{48}$$

(*b*) If $G_1=\{(ab)(cd)(ef)\}$pos we can again use *ace.bdf* for s_γ. We thus obtain a second non-primitive group of order 12, viz:

$$(3)\qquad \{(ab)(cd)(ef)\}\text{pos}\ (ace.bdf)=(abcdef)_{12_2}\dagger$$

This is the only group that corresponds to ABC since the negative substitutions which correspond to the most general G_2 do not have their cubes in this

*The foot note in regard to $(abcdef)_{36_2}$ applies also to this group.

†The foot note in regard to $(abcdef)_{36_3}$ applies also to this group.

G_1. For s_β we may take both $ac.bd$ and $adbc$. We thus obtain two additional groups of order 24, viz :

(4) $\qquad \{ (ab)(cd)(ef) \}$ pos $(ace.bdf)(ac.bd) = (+abcd)_{24}$

(5) $\qquad \{ (ab)(cd)(ef) \}$ pos $(ace.bdf)(adbc) = (\pm abcd)_{24}$

(c) If $G_1 = (ab.cd.ef)$, s_γ may again equal $ace.bdf$. The two substitutions $ab.cd.ef$ and $ace.bdf$ generate the group. The first interchanges the two cycles of the second and the second interchanges the three cycles of the first. The resulting group must therefore have two as well as three systems of non-primitivity, and hence is found in the former list. All the other three possible groups corresponding to ABC are conjugate to this.

For s_β we may use $ac.bd$, but with $ace.bdf$ this will generate $(ace.bdf)$all. Hence this group is also found in the list of non-primitive groups with two systems of non-primitivity. Hence there is no additional non-primitive group for $G_1 = (ab.cd.ef)$.

(d) If $G_1 = 1$ the second condition of s_γ is satisfied by every substitution. The substitutions that may correspond to ABC must be of the third order and are therefore all conjugate so that we need to consider only one of them. We thus obtain the intransitive group

$$(ace.bdf)\text{cyc}.$$

If we take $ac.bd$ for s_β we obtain an intransitive group corresponding to (ABC)all. If we take $ab.dc.ef$ for s_β we obtain a non-primitive group which is also non-primitive in two systems as is evident. Hence $G_1 = 1$ leads to no new non-primitive group.

We have now examined the entire region through degree six with a view to its non-primitive groups and have found the following

LIST OF NON-PRIMITIVE GROUPS THROUGH DEGREE SIX.

Degree	Order	No.	Group
4	4	1	$(abcd)_4$
		2	$(abcd)$cyc
	8	1	$(abcd)_8$
6	6	1	$(abcdef)_6$
		2	$(abcdef)$cyc
	12	1	$(abcdef)_{12}$
		2	$(abcdef)_{12_2}$
	18	1	$(abcdef)_{18}$
	24	1	$(+abcdef)_{24}$
		2	$(\pm abcdef)_{24}$
		3	$(abcdef)_{24_6}$
	36	1	$(abcdef)_{36}$

	2	$(abcdef)_{36_3}$
48	1	$(abcdef)_{48}$
72	1	$(abcdef)_{72}$

GENERAL REMARKS ON THE CONSTRUCTION OF NON-PRIMITIVE GROUPS.

Let it be required to find the non-primitive groups of degree n, n being a composite positive integer greater than three, and let

$$m_1, m_2, \quad . \quad . \quad . \quad . \quad m_e$$

be all the positive integral factors of n (excepting unity) which satisfy the relation

$$m\alpha \leqq \sqrt{n} \qquad \begin{array}{l} \alpha = 1, 2, \quad . \quad . \quad . \quad e \\ \sqrt{\ } \text{ indicates only the arithmetic root.} \end{array}$$

Hence we may divide n as follows:

No. of Systems	No. of Letters in Each System
m_1	$\frac{n}{m_1}$
m_2	$\frac{n}{m_2}$
.	.
.	.
.	.
m_e	$\frac{n}{m_e}$
$\frac{n}{m_1}$	m_1
$\frac{n}{m_2}$	m_2
.	.
.	.
.	.
$\frac{n}{m_e}$	m_e

Two of these relations will become identical when $m_\alpha = \sqrt{n}$ for some value of α in the series

$$1, 2, \quad . \quad . \quad . \quad e.$$

Otherwise they will all be different. From these we see that the number of different ways of dividing n into systems is odd or even as n is or is not a perfect square.

The work of finding all the non-primitive groups for any one of these divisions into systems, *e. g.* the one which contains m_1 systems, may be resolved into the following steps:

(1) Contruct the groups (the G_1's) which have conjugate transitive constituent groups from each of these systems and are so constituted that their con-

stituents admit of the permutations of some transitive group of degree m_1. The constituent transitive groups are clearly of degree $\frac{n}{m_1}$ unless $G_1=1$. The last case does not need consideration when the order of the transitive group of degree m_1 is not a multiple of n.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from December Number.]

Scholion IV: *In which is expounded on a figure a certain consideration on which Euclid probably thought, in order to establish that Postulate of his as 'per se' evident.*

I premise first: within any acute angle BAX (Fig. 12.) can be drawn from any point X of AX a certain straight XB, which under designated even if obtuse angle R, which only with this acute BAX falls short of two right angles; a certain XB, say I, can be drawn, which at a finite remove meets this AB in a certain point B. For just that I have demonstrated in a Scholion after P. XIII. I premise secondly: these AB, AX (Fig. 25) can be understood as produced into the infinite even to certain points Y, and Z; and likewise the aforesaid XB (into the infinite and itself produced even to a certain point Y) can be understood to be so moved above this AB toward the parts of the point Z, that the angle at the point X toward the parts of the point A is always equal to the certain given obtuse angle R.

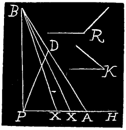


Fig. 12.

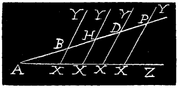


Fig. 25.

I premise thirdly: that Euclidean Postulate would be liable now to no doubt, if the aforesaid XY in this however great motion above the straight AZ cuts always that AY in certain points B, D, H, P, and so successively in other points more remote from this point A.

The reason is evident; since thus any two straights AB, XH lying in the same plane, upon which any straight AX cutting makes two angles toward the

same parts BAX, HXA, less than two right angles, must at length meet toward those parts in one and the same point H.

I premise fourthly : likewise will be no doubt over the truth of the preceding hypothetical assumption, if those later external angles YHD, YDP and so any other succeeding ones, either always are equal to the preceding external angle YBD, or at least always will be not so much less but that any one of them always will be greater than any little designated acute angle K. For, this holding, it is manifest that this XY in that however great motion of its toward the parts of the point Z, never will cease to cut the aforesaid AY; which assuredly (from the preceding note) is sufficient for establishing the controverted postulate.

Solely therefore remains, that a certain adversary may say those external angles at greater and greater distance from that point A may become always less without any determinate limit.

But thence would follow, that that XY in that motion of its above the straight AZ would at length meet AY in a certain point P without any angle with the segment PY, so that indeed a segment of the two straights APY, and XPY would be in this way common.

But this is evidently repugnant to the nature of the straight line. [The possibility that P may be a point at infinity is here overlooked.]

But if indeed to anyone may seem less opportune the obtuse angle at that point X toward the parts of the point A, it may easily be supposed right ; so that indeed (in the motion of the aforesaid XY at angles always right above the straight AZ) more manifestly may appear that the single points of that XY are always moved equably relatively to the basal AZ; and therefore the aforesaid XY cannot go over from a secant into a non-secant of the other indefinite AY, unless either once in some point it precisely touches it, or meets it in some point P, where it has with this AY a common segment PY; each of which I will show contrary to the nature of the straight line in P. XXXIII.

Therefore in accordance with the true idea of the straight line, must that XY, in however great distance of the point X from the point A, always meet in some point this AY. And that this indeed (however small is supposed the acute angle at the point A) is sufficient for demonstrating, against the hypothesis of acute angle, the Euclidean Postulate, will follow from P. XXVII.

[To be Continued.]

THE BOND PROBLEM.

By J. K. ELLWOOD, A. M., Colfax School, Pittsburg, Pennsylvania.

What should an investor pay for one 7 per cent. \$100. bond to run 20 years, interest payable semi-annually, in order to realize 8 per cent. per annum, payable semi-annually ?

Let X=the price paid; $R=4\%$, the semi-annual rate the investor realizes; t=the whole number of interest payments; $r=3\frac{1}{2}\%$, the rate the bond draws semi-annually; $v=\$100$.

Besides the interest, the investor gains $v-x$, which will be due in $\frac{1}{2}t$ years. To liquidate both of these by equal payments requires each semi-annual payment to include the interest (rv) and such portion of the discount ($v-x$) as would, compounded semi-annually at $R\%$, amount to $v-X$ in $\frac{1}{2}t$ years.

Let y be such a sum; then

$y(1+R)^{t-1}$=amount of 1st installment at end of $\frac{1}{2}t$ years.
$y(1+R)^{t-2}$= " " 2nd " " " " "
$y(1+R)$= " " $(t-1)^{th}$ " " " " "
$y(1+R)^0$ = " " t^{th} " " " " "

Hence, $y[(1+R)^{t-1}+(1+R)^{t-2}+\dots\dots(1+R)+1]=v-X.$

Summing the geometrical progression within the brackets, we have

$$y\left[\frac{(1+R)^t-1}{R}\right]=v-X,$$

$$\text{whence } y=\frac{R(v-X)}{(1+R)^t-1}.$$

Therefore each of the t equal payments is

$$vr+\frac{R(v-X)}{(1+R)^t-1},$$

which divided by X gives R.

$$\text{Hence, } vr+\frac{R(v-X)}{(1+R)^t-1}=RX.$$

Solve this equation for X and we have:

$$X=\frac{v(R-r)+vr(1+R)^t}{R(1+R)^t}\dots\dots(A).$$

In the above general equation substitute values from the problem and we have:

$$X=\frac{100(.04-.03\frac{1}{2})+3\frac{1}{2}(1.04)^{40}}{.04(1.04)^{40}}=\frac{\frac{1}{2}+3\frac{1}{2}\times1.04^{40}}{.04\times1.04^{40}}.$$

The easy numerical computations are as follows:

40log $1.04=0.0170333\times40=0.681332$, which corresponds to 4.801.

$$\frac{\frac{1}{2}+3\frac{1}{2}\times4.801}{.04\times4.801}=\frac{17.3035}{.19204}=90.1036.$$

$\therefore$ \$90.1036 is the price to be paid for a 7% \$100 bond, interest payable semi-annually for 20 years, in order to realize 8% per annum, payable semi-annually.

The general equation (A) can be applied to the solution of the quarterly bond. In so applying it "we solve the government problem which confronted the Secretary of the Treasury when he placed the late \$50,000,000 loan on the market." This problem has been admirably solved by Theodore L. DeLand, the distinguished Examiner of the U. S. Civil Service Commission, first by algebraic analysis in THE AMERICAN MATHEMATICAL MONTHLY, and later by using the Calculus of Finite Differences. The latter solution was issued under cover of the *Mathematical Magazine*, January, 1895.

Secretary Carlisle desired to sell 10-year 5% \$100 bonds, interest payable quarterly, at a price that would enable the purchaser to realize 3%, interest payable quarterly.

Using these data, we have $R=\frac{3}{4}\%$, $r=1\frac{1}{4}\%$, $t=40$. Substituting values, equation (A) becomes :

$$x=\frac{100(.00\frac{3}{4}-.01\frac{1}{4})+1\frac{1}{4}(1.0075)^{40}}{.0075\times1.0075^{40}}=\frac{1\frac{1}{4}\times1.0075^{40}-\frac{1}{2}}{.0075\times1.0075^{40}}.$$

40log 1.0075 = 0.0032451 × 40 = 0.129804, which corresponds to 1.34835.

$$\frac{1\frac{1}{4}\times1.34835-\frac{1}{2}}{.0075\times1.34835}=117.223.$$

$\therefore$ \$117.22$\frac{3}{10}$ is a just price for the bonds mentioned.

Problems of this kind may be solved very readily by arithmetic, as follows :

Take the first problem above. The bond yields \$7. per annum, which is 8% of \$87.50. This would be the price if only \$87.50 were to be paid the investor at maturity. But he will receive \$12.50 more, hence he must now give, in addition to the \$87.50, a sum that will in 20 years at 8% compound semi-annual interest amount to \$12.50.

$$\$12.50\div\$4.80102=2.6036.$$

$\therefore$ \$87.50 + \$2.6036 = \$90.1036, the price.

When bonds are bought at a premium, the present value must be *deducted* from the sum that would be the price to be paid provided that sum were to be paid the investor at maturity.

Such problems are readily solved, but the arithmetician requires a very complete compound interest table to cover all cases.

The tables used by brokers give the same prices as those obtained by the methods herein set forth ; but they extend only to 6% bonds to run 60 years.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

The Simson line belonging to one point of intersection of Brocard's Diameter of a triangle with the circumcircle of this triangle, is either parallel or perpendicular to the besector of the angle formed by the side BC of the triangle ABC and the corresponding side $B'C'$ of Brocard's triangle.

Solution by the PROPOSER.

We shall first prove the following lemma :

1. If upon the sides of the triangle ABC are constructed similar isosceles triangles BA_2C, CB_2A, and AC_2B, and if the perpendicular A_2M_a is produced below BC, so that A'_2M_a is equal to A_2M_a then is $AC_2A'_2B_2$ a parallelogram.

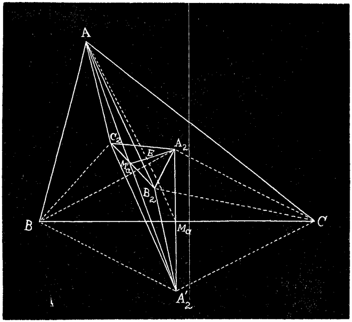


Fig. 1.

$\measuredangle C_2BA'_2=\measuredangle C_2BC+\measuredangle CBA'_2$;

but $\measuredangle CBA'_2=\measuredangle A_2BC$;

therefore $\measuredangle C_2BA'_2=\measuredangle C_2BC+\measuredangle A_2BC$;

but $\measuredangle A_2BC=\measuredangle C_2BA$,

hence $\measuredangle C_1BA'_2=\measuredangle C_2BC+\measuredangle C_2BA-\measuredangle ABC$.

The triangle A_2BM_a is similar to triangle C_2BM_c (since they are right triangles having $\measuredangle A_2BC = \measuredangle C_2BA$).

$$\text{Therefore } A_2B : C_2B = BM_a : BM_c = \frac{a}{2} : \frac{c}{2} = a : c\,;$$

but $\qquad A_2B = A'_2B\,;$

hence $\qquad A'_2B : C_2B = a : c,$

and since the $\measuredangle A'_2BC_2 = \measuredangle ABC$, therefore is triangle A'_2BC_2 similar to triangle ABC. In a similar manner can be proved that the triangle $B_2CA'_2$ is also similar to triangle ABC, and therefore A'_2BC_2 and $B_2CA'_2$ are similar to one another. But $A'_2B = A'_2C$ and consequently are the triangles A'_2BC_2 and $B_2CA'_2$ not only similar but also equal and therefore $B_2A'_2 = C_2A$. In a similar manner can be proved that $AB_2 = C_2A'_2$ or $AC_2A'_2B_2$ is a parallelogram.

2. The triangles ABC and $A_2B_2C_2$ have the same median point E.

Since $AC_2A'_2B_2$ is a parallelogram, the diagonals AA'_2 and A_2C_2 will bisect each other at the point M'_a. $A_2M'_a$ is a median line in the triangle $A_2B_2C_2$ as well as in the triangle $AA_2A'_2$. A second median line in the triangle $AA_2A'_2$ is AM_a (since $A_2M_a = A'_2M_a$); we have, therefore, that $A_2E = 2EM'_a$ and $AE = 2EM_a$. But AM_a is also a median line in the triangle ABC, therefore is E the median point in the triangle ABC as well as in the triangle $A_2B_2C_2$.

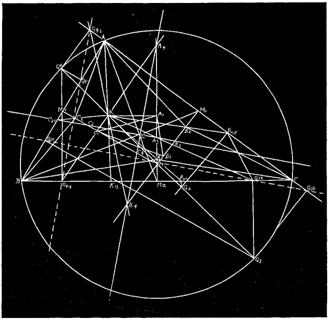


Fig. 2.

A_2, B_2, and C_2 were the vertices of similar isosceles triangles constructed upon the sides of the triangle ABC, and let KA_2, KB_2, and KC_2 meet the sides BC, AC, and AB respectively at $A_{2\alpha}$, $B_{2\beta}$, and $C_{2\gamma}$, then it can be proved that triangle $A_{2\alpha}\ B_{2\beta}\ C_{2\gamma}$ is similar to triangle $A_2\ B_2\ C_2$, the center of similitude being K. If we erect a perpendicular at $A_{2\alpha}$ to BC to meet Brocard's Diameter at Q_2, then, putting for A_1M_a, B_1M_b, their equals KK_a, KK_b respectively, ($A_1B_1C_1$ is Brocard's triangle), we have

$$\frac{A_2M_a}{KK_a}=\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{Q_2M}{Q_2K}.$$

Since the triangles A_2BC and B_2AC are similar, we have

$$\frac{A_2M_a}{B_2M_b}=\frac{M_a\,C}{M_b\,C}=\frac{a}{b}=\frac{A_1M_a}{B_1M_b},$$

or
$$\frac{A_2M_a}{A_1M_a}=\frac{B_2M_b}{B_1M_b}=\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K}.$$

Therefore
$$\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{B_{2\beta}B_2}{B_{2\beta}K}.$$

Similarly we get
$$\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{C_{2\gamma}C_2}{C_{2\gamma}K}=\frac{Q_2M}{Q_2K},$$

or, triangles $A_2B_2C_2$ and $A_{2\alpha}B_{2\beta}C_{2\gamma}$ are similar, and K is the center of similitude. From the equation

$$\frac{B_{2\beta}Q_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K},$$

it follows that $B_{2\beta}Q_2$ is parallel to B_2M, and since B_2M is perpendicular to AC, therefore $B_{2\beta}Q_2$ is also perpendicular to AC, or the perpendicular at $B_{2\beta}$ to AC passes through Q_2. Similarly, the perpendicular at $C_{2\gamma}$ to AB passes through Q_2. If, now, Q_2 is made to coincide with either Q_3 or Q_4, the points of intersection of Brocard's Diameter and the circumcircle of the triangle ABC, the triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ will then degenerate into the straight lines $Q_{3a}\,Q_{3b}\,Q_{3c}$ and $Q_{4a}\,Q_{4b}\,Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4 with respect to the circumcircle of the triangle. The triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$ and $A_4B_4C_4$, which will be parallel to the Simson lines belonging to Q_3 and Q_4; and they will pass through the median point E, for the lines $A_3B_3C_3$ and $A_4B_4C_4$ still have the median point E in common with ABC.

Also, A_3, B_3, C_3 and A_4, B_4, C_4 are on the perpendiculars at the middle points of the respective sides of the triangle ABC. Since the Simson lines to Q_3 and Q_4 correspond to the extremities of a diameter, they are perpendicular to each other, and therefore their parallels $A_3B_3C_3$ and $A_4B_4C_4$ are also perpendicular to each other.

Furthermore, $Q_{3a}M_a=Q_{4a}M_a$,

$$Q_{3a}M_a:M_aK_a=Q_{4a}M_a:M_aK_a\,,$$

$$Q_{3a}M_a:M_aK_a=Q_{3a}A_3:A_3K=A_3M_a:A_3A_1\,,$$

and $$Q_{4a}M_a:M_aK_a=Q_{4a}A_4:A_4K=A_4M_a:A_4A_1\,,$$

or $$A_3M_a:A_3A_1=A_4M_a:A_4A_1\,,$$

whence $\{M_aA_1,\ A_3A_4\}$ is an harmonic range, and $E\{M_aA_1,\ A_3A_4\}$ is an harmonic pencil. Since $\measuredangle A_4EA_3=90^\circ$, EA_3 will bisect the angle A_1EM_a.

Now, in two similar triangles the bisectors of the angles formed by any line in one triangle with the corresponding line in the other triangle are parallel to each other, hence the bisector of the angle formed by A_1E and EM_a, or the line AE, *i. e.*, the line $A_3B_3C_3$,—which is parallel to the Simson line belonging to one of the points of intersection of Brocard's Diameter, and the circumcircle about the triangle ABC,—is parallel to the bisector of the angle formed by B_1, C_1, and BC. (For particulars I can refer to my Geometrical Treatment of curves which are isogonal conjugate to a straight line with respect to a triangle, published by Leach, Shewell and Sanborn, New York.)

An excellent solution of this problem was also received from Professor G. B. M. Zerr.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

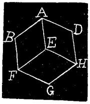
36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsalvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

III. Solution by the PROPOSER.

The $\triangle$ BDE has each side$=\sqrt{2}a$, hence the radius of its circumscribed circle$=\frac{1}{3}\sqrt{6}a$. Hence the distance of A to the plane of $BDE=\frac{1}{3}\sqrt{3}a$. Take the origin at the center of the cube and the line AG as the axis of Z. The revolution will bring each line of the gauche hexagon $EHDCBF$ into either

the position of DH or BF. The equations of DH are $x=\frac{1}{2}\sqrt{2}$, $y=-\sqrt{2z}$, and the equations of BF are $x=-\frac{1}{2}\sqrt{2}$, $y=-\sqrt{2z}$. In either case $x^2=\frac{1}{2}$ and $y^2=2z^2$ and $x^2+y^2=2z^2+\frac{a^2}{2}$ which is the equation of the surface generated by the gauche hexagon $EHDCBF$. This surface could also be generated by the hyperbola $x^2-2z^2+\frac{1}{2}$. Hence the volume of the hyperboloid of one nappe generated $=\int \pi x^2 dz$, the upper limit being $\frac{1}{6}\sqrt{3a}$ and the lower limit $-\frac{1}{6}\sqrt{3a}$. This integral is $\frac{5}{27}\pi\sqrt{3a^3}$.



The lines AB, AE, and AD generate a cone, radius $=\frac{1}{3}\sqrt{6a}$, altitude $=\frac{1}{3}\sqrt{3a}$, volume $=\frac{2}{27}\pi\sqrt{3a^3}$.

The lines GF, GH, and GC generate another cone of the same size.

The sum of the volumes of the three solids $=\frac{1}{3}\pi\sqrt{3a^3}=1.8138a^3$.

[NOTE.—This solution by the Proposer is fuller than that given in the November number, and is published because several of our contributors failed to comprehend the abbreviated solution previously published. Prof. Whitaker asserts that the solution by Dr. Zerr in the September-October number is incorrect, while the latter says he does not as yet see Prof. Whitaker's hyperboloid. The above seems to be correct, but we shall be glad to have the criticisms of other contributors.—EDITOR.]

43. Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.

$$\text{Prove that} \int_2^1 \frac{x^{a-1}+x^{-a}}{1+x}\frac{dx}{x}=\log(\tan\frac{a\pi}{2}), \text{ when } a>0 \text{ and } <1.$$

[*Williamson's Integral Calculus, p. 154.*]

Comment by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

There seems to be an error in No. 43, as I find the following in my copy of *Williamson* :

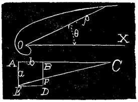
$$\int_0^1 \frac{x^{a-1}+x^{a}}{1+x}\frac{dx}{\log x},$$

which gives the required result.

[In *Williamson's Integral Calculus*, edition of 1891, the problem is given as published, but the mistake has doubtless been corrected in the later edition.—EDITOR.]

44. Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.

Find the equation of a curve in which $\rho=f(\theta)$, in which ρ is equal to BC, an intercept of any secant drawn from the corner E of the rectangle $AEDB$, and prolonged to cut AB prolonged in C. Let equal increments of θ be proportional to the equal incrementsof DB as divided by the secant EF, θ being zero when EC coincides with ED, and $\theta=2\pi$ when EF passes through B. Determine the asymptotes.



I. Solution by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Referring to the diagram given by the Proposer of this problem, July-Au-

gust (1895) MONTHLY, we have from the similar triangles FBC and FDE the following proportions: $BC : DE :: BF : DF$, or $\rho : b :: 2\pi - \theta : \theta$.

$\therefore$ $(\rho+b)\theta = 2\pi b$......(1), which is the polar equation of *The Thistle of Scotland*, adopting the suggestion of Prof. MacCord.

Since $\rho^2(d\theta/d\rho) = -[(2\pi-\theta)^2/2\pi]b$, there is a *rectilinear asymptote* parallel to the initial line and at a distance $2\pi b$ *above* it. Making $\theta = \infty$, we have from (1) the equation $\rho = -b$; and this equation characterizes an *asymptotic circle* of radius b, or a *circular asymptote* of same radius, of the curve.

NOTE.—The derivation of (1) can be affected in, at least, *three* different ways; and, according to the conditions of the problem, (1) may also be written

$$(\rho+b)\theta = ab......(2).$$

II. Solution by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O. Sebastopol, California.

From figure given $\dfrac{BC}{ED} = \dfrac{BF}{DF} = \dfrac{BD-DF}{DF} = \dfrac{BD}{DF} - 1$,

or $\dfrac{\rho}{b} = \dfrac{2\pi}{\theta} - 1$; $\rho = \dfrac{2b\pi}{\theta} - b$, the equation of the curve.

When $\theta = 0$, $\rho = \infty$, and subtangent $= -2b\pi$.

The curve has, therefore, an asymptote parallel to OX at a distance above it, $2b\pi$, the circumference of a circle with radius AB.

The curve is concave toward the pole and intersects the axis perpendicularly and at a distance b to the left of the pole.

Elaborately solved by *O. W. ANTHONY*, and *C. W. M. BLACK*.

ERRATA.—On page 363, of last issue, line 4. omit $\sqrt{3}$ in the numerator of the second term; line 9, in the numerator, for "(a^2+x^2)" read (a^2-x^2); line 11, in the denominator of the second term, for "4" read 4^2; line 14, for "+" read $=$, before the last expression; page 364, line 15, for "of" read *to*; line 17, insert comma after "length"; line 17, for "$2n$" read 2π; line 18, for "π_2" read π^2; on same page, problem No. "43" should be No. 42; page 365, line 1, for "$z^{n\ 2}$" read z^{n-1}; and in line 2, of solution III., for "n^2+y^2" in the exponent, read x^2+y^2.

PROBLEMS.

51. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustrum.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area.

EDITORIALS.

In this issue will be found a bill with the amount due us to the end of 1896 marked thereon.

The great work of preparing the list of contributors and the index for Vol. II. is to be credited to Editor Colaw.

Prof. G. H. Harvill is now permanently located at Athens, Texas, from which place the *Mathematical Messenger* will be issued.

Persons wishing to discontinue their subscriptions to the MONTHLY, and who are not in arrears, should return this number with their names written upon the wrapper.

Mr. John McDowell of Philadelphia, writes as follows: "Find enclosed three dollars, being amount of subscription for your valuable journal, THE AMERICAN MATHEMATICAL MONTHLY, for '96."

This number of the MONTHLY has been cut short in order that we may catch up in its publication. We shall cut the February number some also. The March number will contain the regular departments again.

Prof. P. S. Berg, Larimore, North Dakota, writes, "Enclosed find three dollars as my subscription to THE AMERICAN MATHEMATICAL MONTHLY. . . . I should not be without it if the subscription price were five dollars."

Dr. G. A. Miller, Leipzig, Germany, writing in reference to the MONTHLY, says, "When I return I hope to be able to do much more towards aiding such efforts towards advancing the cause of mathematics in the United States. You are doing a great work. I hope you will not be discouraged in it."

Our valued contributor, Dr. Alexander Macfarlane, has an article on Quaternions in *Science* of January 17th. He has also prepared the article on Vector Analysis and Quaternions in *Higher Mathematics for Engineering Colleges*, a work edited by Drs. Mansfield Merriman and Robert S. Woodward, and which is expected to be ready in July.

Dr. G. B. M. Zerr, of Texarkana College, says, in a letter of January 7th, "I will remit subscription for '96 in a few weeks. I will remit $3.00 and am willing to pay $5.00 if necessary. I find myself very much benefited by the excellent solutions and excellent papers that appear in each number of the MONTHLY. Do not allow its publication to cease, rather raise the subscription price, I am satisfied the subscribers will stand by you."

NOTES.

Drs. Fisher and Schwatt's translation of Dr. H. Durège's Elements of the Theory of Functions is now ready.

Alexander Macmillan, the younger of the two brothers of the firm Macmillan & Co., died in England on January 25.

THE LOBACHEVSKI PRIZE.

On May 1, 1895, the Lobachévski Fund had reached, beyond all expenses, 8840 roubles, 95 kopeks.

This sum permits the accomplishment of the double aim of the committee: to found an international prize for research in geometry, especially non-Euclidean geometry, and to erect a bust of the celebrated scientist.

The prize, 500 roubles, will be adjudged every three years to the best works or memoirs on geometry, especially non-Euclidean geometry.

The prize will be given for works printed in the Russian, French, German, English, Italian, or Latin, sent to the Physico-Mathematical Society of Kazán by the authors, published during the six years which precede the adjudication of the prize. Works to compete must be sent to the Society at the latest one year before the day of award, October 22 old style (November 3).

The first prize will be adjudged October 22 (November 3), 1897.

To award the prize, the Society will form a commission to choose judges among Russian or foreign scientists.

The work of the judges (reporters) will be recompensed by medals of gold, bearing the name of Lobachévski.

As a fixed capital to found this prize, 6000 roubles were invested.

Of the sum collected, an additional 2000 roubles goes to share the expense of erecting a bust of Lobachévski in the park bearing his name in front of the University edifice in Kazán, the remainder of the cost to be borne by the Municipal Council.

A special committee, consisting of representatives of the Municipal Council and of the Physico-Mathematical Society, has made a contract with Mlle. Dillon, who engages for 3000 roubles to furnish a bronze bust of Lobachévski, to be placed on a granite pedestal, the height of the monument to exceed 3 mèters.

It is hoped to unveil the bust between the 15th and 25th of September, 1896.

This 'fête mathématique' will follow the 'congrès des savants russes naturalistes et mathématiciens' at Kiev from 1st to 12th of September, 1896, and be during the grand Russian Exposition artistic and industrial at Nijny-Novgorod in the summer and autumn of 1896. Foreigners in any way identified with the name of Lobachévski are invited to the fête, and such as accept will be the guests of the city and University of Kazán.

For a second bust of Lobachévski to be placed in the Assembly Hall of the

University, 200 roubles have been given from the Lobachévski fund, the remainder of the cost to be borne by the professors of the University.

The residue of the sum already collected (640 r. 95 k.) will be added to the fixed capital. The augmentation of the capital will permit of a new edition of Lobachévski's works in a few years, the first volume of the Kazán edition having already become rare (out of print).

The Physico-Mathematical Society of Kazán has already received a large number of works and memoirs relating to Lobachévski and non-Euclidean geometry, and now having added its own collection of the printed and manuscript works of Lobachévski, the Society has inaugurated a separate library under the name *Bibliotheca Lobachévskiana*. It is hoped that in time this library will collect all the literature of non-Euclidean geometry and be an indispensable aid to those engaged in its development.

All writers on this fecund subject are begged to send to this library copies of their works.

Alas! That the Mathematico-physical Society of Hungary, a country having an equal claim to all the honors of the non-Euclidean geometry through the genius of Bolyai János, should have been content with placing in 1894 a monumental stone on his long neglected grave in Maros-Vásárhely!

GEORGE BRUCE HALSTED.

Austin, Texas.

THE UNIVERSITY OF CHICAGO: SUMMER, 1896.

The following mathematical courses will be offered: By Professor *Moore*, Theory of numbers, Differential equations (with introduction to Lie's continuous transformation groups); by Professor *Bolza*, Theory of substitutions, Theory of functions of a complex variable; by Professor *Miller*, of the University of Indiana, Analytical geometry of three dimensions; by Dr. *Young*, Conferences on mathematical pedagogy, Theory of equations, College algebra; by Mr. *Slaught*, Advanced integral calculus, Introductory course in differential and integral calculus; and by Mr. *Baker*, Analytical geometry of the plane. The pedagogical conferences are two hours weekly for six weeks and the other courses are four or five hours weekly for twelve weeks from July 1, 1896. Those who expect to work in mathematics in the University of Chicago during the coming summer as well as those who desire further information are requested to communicate with Professor Moore.

BOOKS AND PERIODICALS.

Elementary Mensuration. By F. H. Stevens, M. A., Formerly Scholar of Queen's College, Oxford; A Master of the Military Side, Clifton College. 12mo. cloth, 243 pp. Price, 90 cents, net. New York: Macmillan & Co.

This text-book of Elementary Mensuration is divided into two parts. The first part provides for those students whose knowledge of Geometry is confined to Euclid's First Book, and Algebra to the meaning of the simplest symbols. In the second part more difficult questions are offered to students who have mastered the Sixth Book of Euclid, have attained some facility in ordinary Algebraical methods as far as the Binomial Theorem and have made a beginning with Trigonometry.

Under each rule is given an illustrative solution neatly worked out, and proofs of formulæ have been given or indicated whenever they seemed likely to be intelligent to the learner. The book is in every way worthy of the consideration of teachers who are needing a good elementary text on Mensuration. B. F. F.

Problems in Differential Calculus Supplementary to a Treatise on Differential Calculus. By W. E. Byerly, Ph. D., Professor of Mathematics in Harvard University. 8vo. cloth. viii and 72 pp. Price, 80 cents. Boston and Chicago: Ginn & Co.

An excellent collection of about 350 problems to supplement the author's Treatise on the Differential Calculus. While these problems were especially prepared to use in connection with Dr. Byerly's Calculus they will be found useful wherever the subject is studied. B. F. F.

Computation Rules and Logarithms with Tables for other Useful Functions. By Silas W. Holman, Professor of Physics at the Massachusetts Institute of Technology. 8vo. cloth, 73 pp. Price, $1.00, net. New York: Macmillan & Co.

Besides a Table of Five Place Logarithms containing an abbreviated Table for One and Two Place Numbers, a table for five place numbers from 1.0 to 1.1, avoiding interpolation, a table for all four place numbers with interpolation tables for the fifth place; a table of logarithms of sines, cosines, tangents, and cotangents to four places; and a table of logarithms of sines, cosines, tangents, and cotangents to five places; there is also a four place logarithm table of numbers from 1 to 10; a table of square roots and squares of numbers from 1 to 100; a table of reciprocals of numbers from 1 to 1000; a table of slide wire ratios; a table of natural sines, cosines, tangents, and cotangents, and a number of tables of mathematical constants.

A very useful book for the practical computor. B. F. F.

Algebra for Schools and Colleges. By William Freeland, A. B., Head Master of the Harvard School, New York City. 8vo. cloth, 310 pp. Introduction Price, $1.12. New York: Longmans, Green & Co.

With the exception of two or three instances, the author sets no claim to originality. The book is designed to meet the requirements of those students who present themselves for the maximum courses in Freshman work for students who have advanced through the subject of Quadratics only.

Throughout the course tests for revision have been inserted, and a collection of 500 carefully graded Miscellaneous Examples has been given at the end of the book. The number of examples in the book is 5,200. It is very neatly printed on a good quality of paper. B. F. F.

A Primer of the History of Mathematics. By W. W. Rouse Ball, Fellow and Tutor of Trinity College, Cambridge, England. 12mo. cloth, 162 pp. Price, 65 cents. New York: Macmillan & Co.

This most charming little book ought to be used in all Algebra and Geometry classes in order to awaken early an interest in the History of Mathematics. A few years ago, I gave a short lecture to a class of about 60 students in Algebra, on the Arabic System of Notation. After the lecture, a young man said to me, "Is it possible that Arithmetic and Algebra have come down to us in their present form by a gradual development. I thought they were always as they are now." Were some such work as Mr. Ball's Primer used in our classes in Algebra and Geometry, such dense ignorance concerning one of the greatest departments of human knowledge would not exist. No one having then studied Arthemetic would suppose that the subject sprung from the human mind as perfect as Minerva from the head of Jupiter. B. F. F.

The Elements of Physics. A College Text Book. By Edward L. Nichols and William S. Franklin. In three volumes. Vol. I. Mechanics and Heat. 8vo. cloth, 228 pp. Price, $1.50. New York: Macmillan & Co.

In this valuable treatise on Physics, the authors have not attempted to lift the student over difficulties and set him down in easy places. The work, it appears, is written with a view of giving the student the best possible advantage of the subject. The authors have squarely faced the difficulties of the subject and have, as occasion demanded, used the Calculus rather than encounter a subject by long, laborious and indirect methods avoiding the use of the Calculus. However, the degree of mathematical experience of the undergraduate reader has been kept in view and the various proofs and demonstrations have been given the simplest possible form. The concepts of directed and distributed quantity are briefly treated in Chapter II of Vol. I.

From what we know of the first volume we believe that this Treatise will prove to be the best that has yet appeared in this country. B. F. F.

The Basis. A Monthly Magazine. Devoted to Good Citizenship. Edited by Judge Albion W. Tourgee, Mayville, New York. Price, $1.50 per year.

The Basis for January is a pleasant surprise in its new cover. The leading editorial denounces the retirement of the greenback as an "Epoch-Making Crime." In "A Bystander's Notes", Judge Tourgee treats especially the lack of earnest effort on the part of the colored race for the betterment of their condition. The Mob-Record, the Department of Good Government Clubs and "Today's Thought" are well in evidence. There is a good short story and other characteristic matter. The number speaks well of the new management of *The Basis* and its new home on the Chautauqua Hills.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

The *Review of Reviews* for February contains an article which, in the compass of two pages, makes perhaps the most telling and effective exposure of the recent Turkish massacres that has yet been attempted in the English language. The article is based upon full accounts of the massacres, written on the ground by trustworty and intelligent persons—French, English, American, Turk, Kurd, and Armenian—who were eye-witnesses of the terrible scenes. The article estimates the number of killed in the massacres thus far at 50,000, the property destroyed at $40,000,000, and the number of starving survivors at 350,000.

Elements of the Theory of Functions of a Complex Variable with especial reference to the methods of Riemann. By Dr. H. Durege, late Professor in the University of Prague. Authorized translation from the fourth German Edition. By George Egbert Fisher, M. A., Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania, and Isaac J. Schwatt, Ph. D., Instructor in Mathematics in the University of Pennsylvania. Large 8vo. cloth, 288 pp. Price, $2.50. Philadelphia: G. E. Fisher and I. J. Schwatt.

This valuable work comes to us just in time for notice in this issue of the MONTHLY. From only a cursory examination of it, we do not hesitate to emphasize what was said of it in the last issue. The work will afford a most excellent introduction to the study of the Theory of Functions and the intelligent reading of the larger Treatises—such as Forsyth's.

The mechanical and typographical execution of the book is first class. B. F. F.

The Number Concept. Its Origin and Development. By Levi Leonard Connant, Ph. D., Associate Professor of Mathematics in the Worcester Polytechnic Institute. 8vo. cloth, 218 pp. Price, $2.00. New York: Macmillan & Co.

This work forms a most valuable addition to the literature of mathematics. The first chapter treats on Counting; the second, Number System Limits; the third and fourth, Origin of Number Words; the fifth, Miscellaneous Number Bases; the sixth, The Quinary System; the seventh, The Vigesimal System.

The treatment of these subjects is very interesting and evince careful study and research. B. F. F.

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